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ASSESSMENT OF AMPLITUDE FACTORS OF ASYMPTOTIC EXPANSION AT CRACK TIP IN FLEXOELECTRIC SOLID UNDER MODE I LOADINGS

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Abstract: *The flexoelectric effect is the consequence of the coupling of the large strain gradients and the electric polarization in dielectric materials. The large strain gradients appear near the material defects, especially at the crack tips, where the flexoelectric effect redistributes the stress field and influences the crack behaviour and formation. The flexoelectricity is the size dependent material property, which must be included in the governing equilibrium equations. The consequence of this fact is the more complicated form of the asymptotic solution at the crack tip than the asymptotic solution in the classical elasticity. The asymptotic solution at the crack tip in flexoelectric material contains four amplitude factors in the case of the mode I loading conditions. It is purpose of this contribution to derive the expressions of these amplitude factors as the functions of the stress intensity factor* K_I. *The matched asymptotic expansions method is used to assess the amplitude factors. The crack process zone characterized by the size material parameter* l *is chosen as the boundary layer.*

Keywords: Crack, Flexoelectricity, Matched asymptotic expansions, Amplitude factors, Asymptotic solution.

1. Introduction

The flexoelectric effect is the size dependent material property existing in all dielectric materials, see (Yudin, 2013) or (Zubko, 2013). The coupling of the large strain gradients and the electric polarization causes this effect near the material defects and consequently influences the distribution and evolution of the stress fields. The large strain gradients usually occur near the crack tip and therefore it is relevant to suppose that the crack formation and propagation are affected by the flexoelectricity in the dielectric materials, (Tian, 2022). An interesting finding in orthopedics is the self-repairing and remodeling mechanism of micro-cracks in human bones based on the flexoelectric effect, (Vasquez-Sancho, 2018) and (Nunez-Toldra, 2020). Despite many experimental studies of the flexoelectric effect near the crack tip, the detailed theoretical investigation of fracture process in flexoelectric solids is still missing. The reason for this research imbalance is the difficulty in the obtaining of the adequate solution of the partial differential equations of the fourth order which describe the equilibrium of the flexoelectric body. It is extremely difficult or impossible to find the analytical as well as numerical solution of the fracture problem in the solids with the size dependent and flexoelectric material properties. All these solutions are obtained under the simplified or weakened conditions, (Shu, 1999; Grenzelou, 2008; Tian, 2021) and (Repka, 2018). The strain gradient theory for elastic dielectrics was pioneered by Toupin (1962), the first continuum mechanics theory for flexoelectric solids was proposed by Mindlin (1968). The analytical solution shows that the singularity of stresses accounting the strain gradient effect near the crack tip is $p = -3/2$. This singularity is significantly stronger than their counterparts in the classical fracture mechanics. On the other hand, there is no singularity of Cauchy stresses at the crack tip in the strain gradient elastic theory. It is proportional to

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$p = 1/2$ and its structure is different from the classical stress field controlled by the stress intensity factor K, (Kotoul, 2018; Tian, 2022). Based on the Toupin–Mindlin generalized continuum theory of dipolar gradient elasticity, (Georgiadis, 2003) and (Gourgiotis, 2009) derived the expressions for stress and displacement field at the crack tip in a micro-structured solid under remotely loading (plane strain loading for Mode I and II cracks, anti-plane shear loading for Mode III crack). It is noteworthy that the definition of the J-integral with strain gradient effects is also extended by Georgiadis (2003). Aravas (2009) also developed an asymptotic crack tip solution for a material that obeyed a special form of linear isotropic strain gradient elasticity. Mao (2015) performed an asymptotic analysis of the crack tip field in the flexoelectric solids, which is applied in the analysis discussed in this contribution.

2. Governing equations

It is considered an isotropic dielectric solid Ω with flexoelectricity, whose energy density function $U(\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}, \boldsymbol{P})$ depends on the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, its gradient $\nabla\boldsymbol{\varepsilon}$ and the polarization vector field $\boldsymbol{P}$. The Cauchy stress $\boldsymbol{\sigma}$ and the dipolar stress $\boldsymbol{\tau}$ can be defined in variational manner

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, \tau_{ijk} = \frac{\partial U}{\partial(\partial_i \varepsilon_{jk})} \quad (1)$$

and must satisfy the governing equilibrium equations, Mao (2015),

$$\nabla \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) = 0, \quad (2)$$

$$-\epsilon_0 \nabla^2 \varphi + \nabla \cdot \boldsymbol{P} = 0, \quad (3)$$

$$\boldsymbol{E} + \nabla\varphi = 0, \quad (4)$$

where the body force per volume is omitted, ϵ_0 is permittivity of vacuum, $\boldsymbol{E}$ is electric field and φ is electric potential. The isotropic flexoelectric solid is characterized by the Lamé coefficients λ and μ, the Poisson ratio ν, the length scale parameter l, two flexoelectric constants f_1 and f_2, the dielectric permittivity ϵ and by the dependency of the polarization vector field $\boldsymbol{P}$ on the strain gradients $\nabla\boldsymbol{\varepsilon}$. Consequently, the constitutive equations are given as

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}, \quad (5)$$

$$\boldsymbol{P} = a^{-1}[\boldsymbol{E} - f_1 \nabla \mathrm{Tr}(\boldsymbol{\varepsilon}) - 2 f_2 \nabla \cdot \boldsymbol{\varepsilon}], \quad (6)$$

$$\boldsymbol{\tau} = l^2 \nabla\boldsymbol{\sigma} + f_1\left(\boldsymbol{P}\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{P}\boldsymbol{e}_j\boldsymbol{e}_j\right) + f_2\left(\boldsymbol{e}_i\boldsymbol{e}_i\boldsymbol{P} + \boldsymbol{e}_j\boldsymbol{e}_j\boldsymbol{P}\right) + f_2\left(\boldsymbol{e}_i\boldsymbol{P}\boldsymbol{e}_i + \boldsymbol{e}_j\boldsymbol{P}\boldsymbol{e}_j\right), \quad (7)$$

The basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ represent the basis of Cartesian or polar coordinate system and a is reciprocal susceptibility constant given by the relation

$$a^{-1} = \epsilon - \epsilon_0, \quad (8)$$

The governing equations admit following natural boundary conditions

$$\boldsymbol{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) - \widetilde{\nabla} \cdot (\boldsymbol{n} \cdot \boldsymbol{\tau}) + (\widetilde{\nabla} \cdot \boldsymbol{n})\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\tau}) = \boldsymbol{t} \text{ on } \partial\Omega_t, \quad (9)$$

$$\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{\tau}) = \boldsymbol{r} \text{ on } \partial\Omega_r, \quad (10)$$

$$\boldsymbol{n} \cdot (\epsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \omega \text{ on } \partial\Omega_D, \quad (11)$$

where $\boldsymbol{t}$ is the auxiliary force traction, $\boldsymbol{r}$ is the double force traction, ω is surface charge, $\boldsymbol{n}$ is the external normal to the boundary $\partial\Omega$ and $\widetilde{\nabla} = (I - \boldsymbol{n}\boldsymbol{n}) \cdot \nabla$. It is also practical to introduce non-dimensional parameters

$$\alpha = \frac{f_2}{l\sqrt{a\mu}}, \beta = \frac{f_1 + 2f_2}{l\sqrt{a\mu}} \quad (12)$$

3. Matched asymptotic expansions

The following fracture problem assumes the existence of the crack of the length $2L$ in the dielectric material which is under the mode I mechanical loading conditions. The so-called outer solution is the singular asymptotic solution of this problem at the crack tips characterized by the stress intensity factor K_I. There

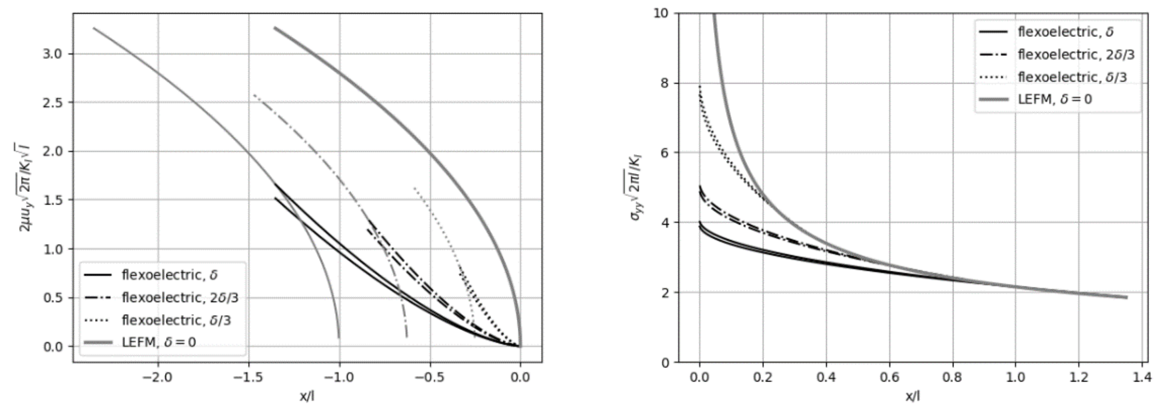

Fig. 1: The crack tip opening displacements u_y (left) and the stress σ_{yy} in front of the crack tip (right) for LELM and flexoelectric material.

are also domains with characteristic dimensions $l \ll L$ at the crack tips, where the strain gradients prevail and couple with electric polarization. The flexoelectric effect must be considered in these small domains, where the dielectric continuum is governed by the equilibrium equations (2)-(4) and the following boundary conditions along the crack faces

$$\boldsymbol{t}(r,\theta) = \boldsymbol{r}(r,\theta) = \omega(0,\theta) = 0 \text{ for } \theta = \pm\pi, \tag{13}$$

where (r,θ) is the polar coordinate system introduced at the crack tip. The boundary conditions (13) and equilibrium equations (2)-(4) allow one to derive the so-called inner asymptotic solution for the Cauchy stress $\boldsymbol{\sigma}$ whose radial and tangential components are as follows

$$\sigma_{rr}^{in} = 2\Gamma_1(\lambda+\mu) + 2\Gamma_2\cos 2\theta + 2\Gamma_3\mu\sin 2\theta + \delta^{1/2}\,R^{1/2}\left\{\left[A_1\left(\frac{5}{2}\lambda + 3\mu\right) + A_4\frac{1}{2}\lambda\right]\cos\frac{1}{2}\theta + A_3\left[\frac{1}{2}\lambda(3l_{12}+5) + 3\mu\right]\cos\frac{3}{2}\theta + 3A_2\mu\cos\frac{5}{2}\theta\right\}, \tag{14}$$

$$\sigma_{r\theta}^{in} = 2\mu(-\Gamma_2\sin 2\theta + \Gamma_3\cos 2\theta) + \delta^{1/2}R^{1/2}\left\{\frac{1}{2}\mu(-A_1+A_4)\sin\frac{1}{2}\theta - 3A_2\mu\sin\frac{5}{2}\theta - \frac{1}{2}A_3\mu(-l_{12}+3)\sin\frac{3}{2}\theta\right\}, \tag{15}$$

where l_{12} is non-dimensional parameter depending on flexoelectric constants. The inner domain represents the crack process zone $r \ll l$, in which the flexoelectric effects are appeared due to the dominancy of the strain gradients. The inner domain can be characterized by the non-dimensional parameter $\delta = l/L \ll 1$ and the scaled coordinate system $(R,\theta) \equiv (r\delta^{-1},\theta)$ allowing the evaluation of the amplitude factors A_1-A_4 as the functions of the stress intensity factor K_I. The strain gradients and dipolar stress $\boldsymbol{\tau}$ decrease to zero value with the increasing distance r from the crack tip. Consequently, the auxiliary force traction (9) converges to the standard traction vector $\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma}$. The double force tractions (10) and surface charge (11) disappear. Hence in the case of the circular shaped remote boundary of the outer domain ($R \to l\delta^{-1} = L$) as well as the near crack tip boundary of the inner domain ($r \to l$), the following equalities can be considered

$$\boldsymbol{t}^{out}\big|_{r=l} \approx \boldsymbol{t}^{in}\big|_{R=l\delta^{-1}=L} \Rightarrow \sigma_{rr}^{out}\big|_{r=l} \approx -\sigma_{rr}^{in}\big|_{R=l\delta^{-1}=L}, \sigma_{r\theta}^{out}\big|_{r=l} \approx -\sigma_{r\theta}^{in}\big|_{R=l\delta^{-1}=L}. \tag{16}$$

The substitution (14) and (15) into (16) leads to the overdetermined system of algebraic equations with unknowns A_1-A_4. This system gives unique solution for A_1, A_3 and A_4. The magnitude of amplitude factor A_2 oscillates between two opposite values

$$A_1 = -\frac{K_I}{(2\pi)^{\frac{1}{2}}}\frac{\lambda-5\mu}{12l\mu(\lambda+\mu)}, A_2 = \pm\frac{K_I}{(2\pi)^{1/2}}\frac{l_{12}(3\lambda+\mu)+5\lambda+3\mu}{12l\mu[l_{12(3\lambda-\mu)}+5\lambda+9\mu]}, \tag{17}$$

$$A_3 = -\frac{K_I}{(2\pi)^{1/2}}\frac{1}{l[l_{12}(3\lambda-\mu)+5\lambda+9\mu]}, A_4 = \frac{K_I}{(2\pi)^{1/2}}\frac{5\lambda+11\mu}{12\mu l(\lambda+\mu)} \tag{18}$$

The parameters Γ_1, Γ_2 and Γ_3 depends on the finite values of the Cauchy stress $\boldsymbol{\sigma}$ at the crack tips. These values are unavailable and must be assessed from the values of $\boldsymbol{\sigma}^{out}$ near the crack tip, e.g. in the distance $r = l$ and $\theta = 0$.

4. Numerical results

It is considered the dielectric material with $\alpha = -0.572$ and $\beta = -0.499$. Figure 1 shows the crack opening displacements u_y and the Cauchy stress σ_{yy} in front of crack tip for varying parameter $\delta, 2\delta/3$ and $\delta/3$, where $\delta = 1 \times 10^{-3}$. The dark lines represent the inner solutions which consider the flexoelectric effect in the crack process zone with dimension l. The top and bottom line of each inner solution correspond to $\pm A_2$, respectively. The minimal influence of the amplitude factor A_2 on the stress and displacement field at the crack tip is obvious. The gray lines represent outer solutions corresponding to the negligible influence of strain gradients and consequently no flexoelectric effects. The LEFM solution for $\delta = 0$ is also depicted as the grey thick line. It can be seen good transition between the outer and the inner solution inside the boundary layers $x/l \sim 1, 2/3$ and $1/3$.

5. Conclusions

The contribution discussed the procedure of the assessment of the amplitude factors appearing in the asymptotic solution of the crack in the flexoelectric solid. The amplitude factors are evaluated under the assumption of the knowledge of the stress intensity factor K_I representing the classical solution of LEFM. There are many ways to redistribute K_I among four amplitude factors appearing in the asymptotic flexoelectric solution and the used matched asymptotic expansion method seems to be applicable tool to get the relevant result

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